Prova: Decomposição de Cholesky (Schott, 4.3)

Seja A matriz não negativa definida, simétrica $m \times m$

para $m=1$, o resultado vale, pois: $A = (a_{11}) = \sqrt{a_{11}} \cdot \sqrt{a_{11}}$, que vale, pois $a_{11} > 0$

Agora, suponha que o resultado vale p/ $(m-1)$. Então:

$$A = \begin{bmatrix} A_{11} & \underset{\sim}{a}_{12} \\ \underset{\sim}{a}_{12}' & a_{22} \end{bmatrix}$$

; com A_{11} matriz simétrica $(m-1) \times (m-1)$

$\underset{\sim}{a}_{12}$ vetor $(m-1) \times 1$

a_{22} escalar.

Como A é n. neg. definida então A_{11} também deve ser não negativa definida. Então existe matriz T triangular inferior de dimensão $(m-1) \times (m-1)$, tal que $A_{11} = TT'$, pois estamos supondo que o resultado vale p/ $(m-1)$.

Agora, precisamos encontrar o vetor $\underset{\sim}{t}_{12}$ e o escalar t_{22} tais que:

$$A = \begin{bmatrix} A_{11} & a_{12} \\ \underset{\sim}{a}_{12}' & a_{22} \end{bmatrix} = LL' = \begin{bmatrix} T & 0 \\ \underset{\sim}{t}_{12}' & t_{22} \end{bmatrix} \begin{bmatrix} T' & \underset{\sim}{t}_{12} \\ 0 & t_{22} \end{bmatrix} = \begin{bmatrix} TT' & T\underset{\sim}{t}_{12} \\ \underset{\sim}{t}_{12}'T' & \underset{\sim}{t}_{12}'\underset{\sim}{t}_{12} + t_{22}^2 \end{bmatrix}$$

Logo, temos:

$$A = \begin{bmatrix} TT' & T\underset{\sim}{t}_{12} \\ \underset{\sim}{t}'_{12}T' & \underset{\sim}{t}'_{12}\underset{\sim}{t}_{12} + t_{22}^2 \end{bmatrix}$$

$$\Rightarrow \begin{cases} \underset{\sim}{a}_{12} = T\underset{\sim}{t}_{12} \longrightarrow \underset{\sim}{t}_{12} = T^{-1}\underset{\sim}{a}_{12} \\ a_{22} = t'_{12}t_{12} + t_{22}^2 \end{cases} \rightsquigarrow t_{22} = a_{22} - t'_{12}t_{12} =$$

$$= a_{22} - a'_{12}(T^{-1})^T \cdot (T^{-1})a_{12}$$

$$= a_{22} - a'_{12}(T')^{-1}(T^{-1})a_{12}$$

$$= a_{22} - a'_{12}(TT')^{-1}a_{12}$$

$$= a_{22} - a_{12}^T A^{-1} a_{12}$$

Matriz 2x2

Exemplo 1: $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$

$$\begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$$

$$l_{11} = \sqrt{a_{11}}$$

$$l_{21} = a_{21}/l_{11} = a_{21}/\sqrt{a_{11}}$$

$$l_{22} = \sqrt{a_{22} - l_{21}^2}.$$

Portanto,

$$L = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \dfrac{a_{21}}{\sqrt{a_{11}}} & \sqrt{a_{22} - l_{21}^2} \end{bmatrix}$$

Matriz 3x3

Calcule a decomposição de Cholesky para:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$$l_{11} = \sqrt{a_{11}}$$

$$l_{21} = a_{21}/\sqrt{a_{11}}$$

$$l_{31} = a_{31}/\sqrt{a_{11}}$$

$i = 2$

$$l_{22} = \sqrt{a_{22} - l_{21}^2}$$

$n = 3;\ i = 2;\ j = 3$

$$l_{32} = \frac{1}{l_{22}}\left(a_{32} - l_{21} l_{31}\right)$$

$i = 3$

$$l_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} l_{3k}}$$

$$= \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$$

Matriz 4x4

$$A = \begin{bmatrix} a_{11} & a_{21} & a_{31} & a_{41} \\ a_{12} & a_{22} & a_{32} & a_{42} \\ a_{13} & a_{23} & a_{33} & a_{43} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{bmatrix}$$

$$\begin{bmatrix} l_{11} & 0 & 0 & 0 \\ l_{21} & l_{22} & 0 & 0 \\ l_{31} & l_{32} & l_{33} & 0 \\ l_{41} & l_{42} & l_{43} & l_{44} \end{bmatrix}$$

$$l_{11} = \sqrt{a_{11}}$$

$$l_{21} = \frac{a_{21}}{\sqrt{a_{11}}}$$

$$l_{31} = \frac{a_{31}}{\sqrt{a_{11}}}$$

$$l_{41} = \frac{a_{41}}{\sqrt{a_{11}}}$$

$i = 2$

$$l_{22} = \sqrt{a_{22} - l_{21}^2}$$

$i < n$ TRUE

$j = 3$

$$l_{32} = \frac{1}{l_{22}}\left(a_{32} - l_{21} l_{31}\right)$$

$$l_{42} = \frac{1}{l_{ii}}\left(a_{42} - l_{21} l_{41}\right)$$

$i = 3$

$$l_{33} = \sqrt{a_{33} - \sum_{k=1}^{k} l_{3k}^2}$$

$$= \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$$

$j = 4$

$$l_{43} = \frac{1}{l_{33}} \sqrt{a_{43} - \sum_{k=1}^{2} l_{3k} l_{4k}}$$

$$= \frac{1}{l_{33}} \sqrt{a_{43} - l_{31} l_{41} - l_{32} l_{42}}$$

$i = 4$

$$l_{44} = \sqrt{a_{44} - \sum_{k=1}^{3} l_{4k}^2}$$

$$= \sqrt{a_{44} - l_{41}^2 - l_{42}^2 - l_{43}^2}$$

Conta feita na aula

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$; é não neg. def.

$L = \begin{bmatrix} l_{11} & 0 \\ l_{21} & l_{22} \end{bmatrix}$

$l_{11} = \sqrt{a_{11}}$.

$n = 2$

$l_{21} = a_{21}/l_{11} = a_{21}/\sqrt{a_{11}}$

$i = 2$

$l_{22} = \sqrt{a_{22} - \sum_{k=1}^{1} l_{2k}^2} = \sqrt{a_{22} - l_{21}^2}$

$$L = \begin{bmatrix} \sqrt{a_{11}} & 0 \\ \dfrac{a_{21}}{\sqrt{a_{11}}} & \sqrt{a_{22} - l_{21}^2} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Conta feita na aula

$$L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$$

$i = \cancel{n}\ 1$

$$l_{11} = \sqrt{a_{11}}$$

$$l_{21} = a_{21}/\cancel{l_{11}} = a_{21}/\sqrt{a_{11}}$$

$$l_{31} = a_{31}/l_{11} = a_{31}/\sqrt{a_{11}}$$

$i = 2$

$$l_{22} = \sqrt{a_{22} - \sum_{k=1}^{1} l_{2k}^2} = \sqrt{a_{22} - l_{21}^2}$$

$j = 3 \ (i=2)$

$$l_{32} = \frac{1}{l_{22}}\left(a_{32} - \sum_{k=1}^{1} l_{2k} \cdot l_{3k}\right)$$

$$l_{32} = \frac{1}{l_{22}}\left(a_{32} - l_{21} \cdot l_{31}\right)$$

$i = 3$

$$l_{33} = \sqrt{a_{33} - \sum_{k=1}^{2} l_{3k}^2} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2}$$